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Dołęga, Maciej ; Féray, Valentin

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# CUMULANTS OF JACK SYMMETRIC FUNCTIONS AND $b$ -CONJECTURE

MACIEJ DOŁĘGA AND VALENTIN FÉRAY

**ABSTRACT.** Goulden and Jackson (1996) introduced, using Jack symmetric functions, some multivariate generating series  $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, 1 + \beta)$  that might be interpreted as a continuous deformation of the generating series of rooted hypermaps. They made the following conjecture: the coefficients of  $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, 1 + \beta)$  in the power-sum basis are polynomials in  $\beta$  with nonnegative integer coefficients (by construction, these coefficients are rational functions in  $\beta$ ).

We prove partially this conjecture, nowadays called *b-conjecture*, by showing that coefficients of  $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, 1 + \beta)$  are polynomials in  $\beta$  with rational coefficients. A key step of the proof is a *strong factorization property* of Jack polynomials when the Jack-deformation parameter  $\alpha$  tends to 0, that may be of independent interest.

## 1. INTRODUCTION

**1.1. Jack symmetric functions.** Jack [Jac71] introduced a family of symmetric polynomials — which are now known as *Jack polynomials*  $J_\pi^{(\alpha)}$  — indexed by a partition and a deformation parameter  $\alpha$ . From the contemporary point of view, probably the main motivation for studying Jack polynomials comes from the fact that they are a special case of the celebrated *Macdonald polynomials* which “*have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics*” [GR05]. Indeed, some surprising features of Jack polynomials [Sta89] have led in the past to the discovery of Macdonald polynomials [Mac95], and Jack polynomials have been regarded as a relatively easy case, which later allowed the understanding of the more difficult case of Macdonald polynomials (the series of papers [LV95, LV97] illustrates this very well). A brief overview of Macdonald polynomials and their relationship to Jack polynomials is given in [GR05]. Jack polynomials are also interesting on their own, for instance in the context of Selberg integrals [Kad97] and in theoretical physics [FJMM02, BH08].

Finally, according to Goulden and Jackson [GJ96], Jack polynomials are also related to hypermap enumeration, via specific multivariate generating functions.

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This relation is still partially a conjecture, and the main result of this paper is a step towards its resolution.

**1.2.  $b$ -conjecture and our main result.** In the following,  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  are three disjoint infinite alphabets. Let  $J_\lambda^{(\alpha)}(\mathbf{x})$  (resp.  $J_\lambda^{(\alpha)}(\mathbf{y})$ ,  $J_\lambda^{(\alpha)}(\mathbf{z})$ ) be the Jack symmetric function in  $\mathbf{x}$  (resp.  $\mathbf{y}$ ,  $\mathbf{z}$ ) indexed by a partition  $\lambda$ . Let us denote by  $\text{hook}_\alpha(\lambda)$  and  $\text{hook}'_\alpha(\lambda)$  the  $\alpha$  hook-polynomials (these are combinatorial factors that appear often in Jack polynomial theory; see Section 2.1 for the definition). We also use the notation  $\mathcal{P}$  for the set of all integer partitions and  $|\lambda|$  for the size of a partition  $\lambda$ . Goulden and Jackson [GJ96] defined a family of coefficients  $h_{\mu,\nu}^\tau(\alpha - 1)$  by the following formal series identity:

$$(1) \quad \log \left( \sum_{\lambda \in \mathcal{P}} \frac{J_\lambda^{(\alpha)}(\mathbf{x}) J_\lambda^{(\alpha)}(\mathbf{y}) J_\lambda^{(\alpha)}(\mathbf{z}) t^{|\lambda|}}{\text{hook}_\alpha(\lambda) \text{hook}'_\alpha(\lambda)} \right) \\ = \sum_{n \geq 1} \frac{t^n}{\alpha n} \left( \sum_{\mu, \nu, \tau \vdash n} h_{\mu,\nu}^\tau(\alpha - 1) p_\mu(\mathbf{x}) p_\nu(\mathbf{y}) p_\tau(\mathbf{z}) \right),$$

where  $\mu, \nu, \tau \vdash n$  means that  $\mu$ ,  $\nu$  and  $\tau$  are three partitions of  $n$  and  $p_\mu$  is the power-sum symmetric function associated with  $\mu$ .

This rather involved definition is motivated by the following combinatorial interpretations for particular values of  $\alpha$ ; see [GJ96, Section 1.1] and references therein.

- In the case  $\alpha = 1$ , the quantity  $h_{\mu,\nu}^\tau(0)$  enumerates connected hypergraphs embedded into oriented surfaces with vertex-, edge- and face-degree distributions given by  $\mu$ ,  $\nu$  and  $\tau$ , respectively.
- In the case  $\alpha = 2$ , the quantity  $h_{\mu,\nu}^\tau(1)$  enumerates connected hypergraphs embedded into non-oriented surfaces with the same degree conditions.

Connected hypergraphs embedded into surfaces are usually called *maps* and are a classical topic in enumerative combinatorics related to the computation of matrix integrals and to the study of moduli spaces of curves, as explained in detail in the book [LZ04]. The logarithm in Eq. (1) is present because we only want to count connected objects.

Note that  $h_{\mu,\nu}^\tau(\alpha - 1)$  depends on the parameter  $\alpha$ , and describing it as a function of  $\beta := \alpha - 1$  might seem be artificial. However, this shift seems to be the right one for finding a combinatorial interpretation of  $h_{\mu,\nu}^\tau(\beta)$ , as suggested by Goulden and Jackson [GJ96] in the following conjecture.

**Conjecture 1.1** ( $b$ -conjecture). *For all partitions  $\tau, \mu, \nu \vdash n \geq 1$ , the quantity  $h_{\mu,\nu}^\tau(\beta)$  is a polynomial in  $\beta$  with nonnegative, integer coefficients. Moreover, there exists a statistics  $\eta$  on maps such that*

$$(2) \quad h_{\mu,\nu}^\tau(\beta) = \sum_{\mathcal{M}} \beta^{\eta(\mathcal{M})},$$

where the summation index runs over all rooted, bipartite maps  $\mathcal{M}$  with face distribution  $\tau$ , black vertex distribution  $\mu$  and white vertex distribution  $\nu$ , and  $\eta(\mathcal{M})$  is a nonnegative integer equals to 0 if and only if  $\mathcal{M}$  is orientable.

This conjecture is still open. The thesis of La Croix [La 09] gives a number of evidences for it, and gives a good account of what is known so far. In particular, some constructions for a candidate statistics  $\eta$  have been given, establishing particular cases of the conjecture [BJ07, La 09, KV16]. However, there is not much known about the structure of  $h_{\mu,\nu}^\tau(\beta)$  for arbitrary partitions  $\tau, \mu, \nu \vdash n$ . Strictly from the construction they are rational functions in  $\beta$  with rational coefficients. Our main result in this paper is a proof of the following polynomiality result for  $h_{\mu,\nu}^\tau(\beta)$  for all partitions  $\tau, \mu, \nu \vdash n \geq 1$ .

**Theorem 1.2.** *For all partitions  $\tau, \mu, \nu \vdash n \geq 1$ , the quantity  $h_{\mu,\nu}^\tau(\beta)$  is a polynomial in  $\beta$  of degree  $2 + n - \ell(\tau) - \ell(\mu) - \ell(\nu)$  with rational coefficients.*

Unfortunately, the nonnegativity and the integrality of the coefficients seem out of reach with our approach. However, the polynomiality could be useful in the investigation of 1.1. In particular, the first author has recently found a combinatorial description of the top-degree part of  $h_{\mu,\nu}^{(n)}(\beta)$  [Do16b]. Theorem 1.2 is one of the ingredients of the proof.

**1.3. Strong factorization of Jack polynomials.** A key step in our proof is a *strong factorization property* for Jack polynomials when  $\alpha$  tends to zero. To state it, let us introduce a few notations. If  $\lambda^1$  and  $\lambda^2$  are partitions, we denote by  $\lambda^1 \oplus \lambda^2$  their entry-wise sum. If  $\lambda^1, \dots, \lambda^r$  are partitions and  $I$  a subset of  $[r] := \{1, \dots, r\}$ , then we denote

$$\lambda^I := \bigoplus_{i \in I} \lambda^i.$$

**Theorem 1.3.** *Let  $\lambda^1, \dots, \lambda^r$  be partitions. Then*

$$(3) \quad \prod_{I \subseteq [r]} (J_{\lambda^I}^{(\alpha)})^{(-1)^{|I|}} = 1 + O(\alpha^{r-1}),$$

when  $\alpha \rightarrow 0$ .

Here, to give sense to the  $O$  notation, we consider  $J_{\lambda^I}^{(\alpha)}$  as a function of a real variable  $\alpha$ . Since all involved quantities are rational functions, it is however also possible to think of  $\alpha$  as a formal parameter; see Definition 3.2 for the meaning of  $O(\alpha^{r-1})$  in this context.

The exponent  $(-1)^{|I|}$  may be a bit disturbing so let us unpack the notation for small values of  $r$ .

- For  $r = 2$ , Eq. (3) writes as

$$\frac{J_{\lambda^1 \oplus \lambda^2}^{(\alpha)}}{J_{\lambda^1}^{(\alpha)} J_{\lambda^2}^{(\alpha)}} = 1 + O(\alpha).$$

Since we have rational functions, it is equivalent to say that, for  $\alpha = 0$ , one has the factorization property  $J_{\lambda^1 \oplus \lambda^2}^{(0)} = J_{\lambda^1}^{(0)} J_{\lambda^2}^{(0)}$ . This is indeed true and follows from an explicit expression for  $J_{\lambda}^{(0)}$  given by Stanley; see [Sta89,

Proposition 7.6] or Eq. (12) in this paper. Thus, in this case, our theorem does not give anything new.

- For  $r = 3$ , Eq. (3) writes as

$$\frac{J_{\lambda^1 \oplus \lambda^2 \oplus \lambda^3}^{(\alpha)} J_{\lambda^1}^{(\alpha)} J_{\lambda^2}^{(\alpha)} J_{\lambda^3}^{(\alpha)}}{J_{\lambda^1 \oplus \lambda^2}^{(\alpha)} J_{\lambda^1 \oplus \lambda^3}^{(\alpha)} J_{\lambda^2 \oplus \lambda^3}^{(\alpha)}} = 1 + O(\alpha^2).$$

Using the above case  $r = 2$ , it is easily seen that the left-hand side is  $1 + O(\alpha)$ . However, our theorem says more and asserts that it is  $1 + O(\alpha^2)$ , which is not trivial at all.

This explains the terminology *strong factorization property*.

The theorem has an equivalent form that uses the notion of *cumulants of Jack polynomials* — see Section 3 for comments on the terminology. For partitions  $\lambda^1, \dots, \lambda^r$ , we denote

$$\kappa^J(\lambda^1, \dots, \lambda^r) = \sum_{\pi \in \mathcal{P}([r])} \mu(\pi, \{H\}) \prod_{B \in \pi} J_{\lambda_B}.$$

Here, the sum is taken over set partitions  $\pi$  of  $[r]$  and  $\mu$  stands for the Möbius function of the set partition lattice; the Reader not familiar with these concepts can have a look to Section 2.4. For example

$$\begin{aligned} \kappa^J(\lambda^1, \lambda^2) &= J_{\lambda^1 \oplus \lambda^2}^{(\alpha)} - J_{\lambda^1}^{(\alpha)} J_{\lambda^2}^{(\alpha)}, \\ \kappa^J(\lambda^1, \lambda^2, \lambda^3) &= J_{\lambda^1 \oplus \lambda^2 \oplus \lambda^3}^{(\alpha)} - J_{\lambda^1}^{(\alpha)} J_{\lambda^2 \oplus \lambda^3}^{(\alpha)} \\ &\quad - J_{\lambda^2}^{(\alpha)} J_{\lambda^1 \oplus \lambda^3}^{(\alpha)} - J_{\lambda^3}^{(\alpha)} J_{\lambda^1 \oplus \lambda^2}^{(\alpha)} + 2J_{\lambda^1}^{(\alpha)} J_{\lambda^2}^{(\alpha)} J_{\lambda^3}^{(\alpha)}. \end{aligned}$$

We then have the following estimate for cumulants of Jack polynomials

**Theorem 1.4.** *For any partitions  $\lambda^1, \dots, \lambda^r$ , one has*

$$(4) \quad \kappa^J(\lambda^1, \dots, \lambda^r) = O(\alpha^{r-1}).$$

Theorem 1.4 is in fact equivalent to Eq. (3), as shown (in a more general setting) by Proposition 3.3 (we need here the fact that  $J_\lambda$  has a non-zero limit when  $\alpha$  tends to 0 [Sta89, Proposition 7.6]; this ensures that  $J_\lambda = O(1)$  and  $J_\lambda^{-1} = O(1)$ ). We prove Theorem 1.4 in Section 4.

We noticed, using computer simulations, that a similar property seems to hold for Macdonald polynomials  $J_\lambda^{(q,t)}$ . Unfortunately, we were unable to prove it and we state it here as a conjecture. Similarly to the Jack case, we define

$$\kappa^M(\lambda^1, \dots, \lambda^r) = \sum_{\pi \in \mathcal{P}([r])} \mu(\pi, \{H\}) \prod_{B \in \pi} J_{\lambda_B}^{(q,t)}.$$

**Conjecture 1.5.** *For any partitions  $\lambda^1, \dots, \lambda^r$ , one has:*

- the strong factorization property of Macdonald polynomials when  $q$  goes to 1, i.e.

$$(5) \quad \prod_{I \subset [r]} (J_{\lambda^I}^{(q,t)})^{(-1)^{|I|}} = 1 + O\left((q-1)^{r-1}\right),$$

when  $q \rightarrow 1$ ;

- the following estimates on cumulants of Macdonald polynomials

$$(6) \quad \kappa^M(\lambda^1, \dots, \lambda^r) = O\left((q-1)^{r-1}\right),$$

when  $q \rightarrow 1$ .

As in the Jack case, the above items are equivalent by Proposition 3.3. Note that the case  $r = 2$  of both items says that

$$J_{\lambda^1 \oplus \lambda^2}^{(1,t)} = J_{\lambda^1}^{(1,t)} J_{\lambda^2}^{(1,t)},$$

which follows from the explicit expression for  $J_{\lambda}^{(1,t)}$  given in [Mac95, Chapter VI, Remark (8.4), item (iii)]. Moreover, we mention that Conjecture 1.5 implies Theorem 1.4 as a special case by substitution  $q = t^\alpha$  and taking a limit  $t \rightarrow 1$  since one has (see [Mac95, Chapter VI, Eq. (10.23)]):

$$\lim_{t \rightarrow 1} (1-t)^{-|\lambda|} J_{\lambda}^{(t^\alpha, t)}(x) = J_{\lambda}^{(\alpha)}(x).$$

*Note added in revision:* After submission of the current paper, the first author has found a proof of Conjecture 1.5; see [Dof16a].

**1.4. Related problems.** We finish this section mentioning two similar problems. First, a very similar conjecture to Conjecture 1.1 (without logarithm in Equation (1)) was also stated by Goulden and Jackson [GJ96]. The series obtained in this way is conjecturally a multivariate generating function of *matchings*, where the exponent of  $\beta$  is some combinatorial integer-valued statistics. The conjecture is still open, while some special cases have been solved by Goulden and Jackson in their original article [GJ96] and recently by Kanunnikov and Vassilieva [KV16]. The polynomiality result for the coefficients of this series was proven by the authors of this paper in [DF16] and is significant in the current work. Indeed, together with a simple argument given in Section 2.3, it reduces the proof of Theorem 1.2 to checking that there is no singularity in  $\alpha = 0$ .

The second related problem is the investigation of *Jack characters*, that is suitably normalized coefficients of the power-sum expansion of Jack polynomials. In a series of papers [Las08, Las09] Lassalle stated some polynomiality and positivity conjectures suggesting that a combinatorial description of these objects might exist. Although these conjectures are not fully resolved, it was proven by us together with Śniady [DFS14] and by Śniady [Śni15] that in some special cases indeed, such combinatorial setup exists. Moreover, similarly to Conjecture 1.1, these special cases involve hypermaps and some statistics that “measures their non-orientability”.

We cannot resist to state that there must be a deep connection between all these problems, and understanding it would be of great interest.

**1.5. Organization of the paper.** We describe all necessary definitions and background in Section 2, and in Section 3 we discuss cumulants and their relation with strong factorization. Section 4 is devoted to the proof of the strong factorization property of Jack polynomials, while Section 5 presents the proof of the main result, that is the polynomiality in  $b$ -conjecture.

## 2. PRELIMINARIES

**2.1. Partitions.** We call  $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_l)$  a *partition* of  $n$  if it is a weakly decreasing sequence of positive integers such that  $\lambda_1 + \lambda_2 + \dots + \lambda_l = n$ . Then  $n$  is called *the size* of  $\lambda$  while  $l$  is *its length*. As usual, we use the notation  $\lambda \vdash n$ , or  $|\lambda| = n$ , and  $\ell(\lambda) = l$ . We denote the set of partitions of  $n$  by  $\mathcal{Y}_n$  and we define a partial order on  $\mathcal{Y}_n$ , called *dominance order*, in the following way:

$$\lambda \leq \mu \iff \sum_{i \leq j} \lambda_i \leq \sum_{i \leq j} \mu_i \text{ for any positive integer } j.$$

For any two partitions  $\lambda \in \mathcal{Y}_n$  and  $\mu \in \mathcal{Y}_m$  we can construct two new partitions  $\lambda \oplus \mu, \lambda \cup \mu \in \mathcal{Y}_{n+m}$ , where  $\lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$  and  $\lambda \cup \mu$  is obtained by merging parts of  $\lambda$  and  $\mu$  and ordering them in a decreasing fashion. Moreover, there exists a canonical involution on the set  $\mathcal{Y}_n$ , which associates with a partition  $\lambda$  its *conjugate partition*  $\lambda^t$ . By definition, the  $j$ -th part  $\lambda_j^t$  of the conjugate partition is the number of positive integers  $i$  such that  $\lambda_i \geq j$ . Notice that for any two partitions  $\lambda, \mu$ , we have  $(\lambda \cup \mu)^t = \lambda^t \oplus \mu^t$ . A partition  $\lambda$  is identified with some geometric object, called *Young diagram* drawn in French convention, that can be defined as follows:

$$\lambda = \{(i, j) : 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell(\lambda)\}.$$

For any box  $\square := (i, j) \in \lambda$  from Young diagram we define its *arm-length* by  $a(\square) := \lambda_j - i$  and its *leg-length* by  $\ell(\square) := \lambda_i^t - j$  (the same definitions as in [Mac95, Chapter I]), see Fig. 1.

There are many combinatorial quantities associated with partitions that we will use extensively through this paper, so let us define them. First, set

$$(7) \quad z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!,$$

where  $m_i(\lambda)$  denotes the number of parts of  $\lambda$  equal to  $i$ . We also define  $\alpha$ -hook polynomials  $\text{hook}_\alpha(\lambda)$  and  $\text{hook}'_\alpha(\lambda)$  by the following equations:

$$(8) \quad \text{hook}_\alpha(\lambda) := \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + 1),$$

$$(9) \quad \text{hook}'_\alpha(\lambda) := \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + \alpha).$$

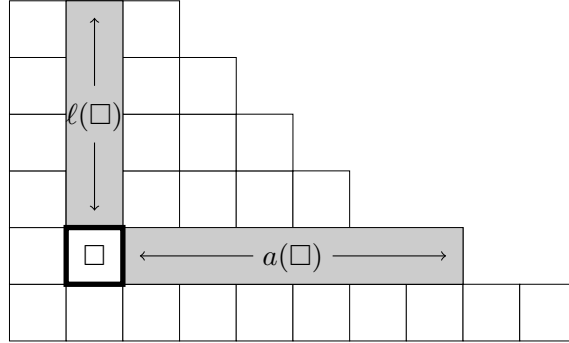


Figure 1. Arm and leg length of boxes in Young diagrams. Above Young diagram corresponds to partition  $\lambda = (10, 8, 6, 5, 4, 3)$ .

Finally, we consider a partition binomial given by

$$(10) \quad b(\lambda) := \sum_i \binom{\lambda_i}{2}.$$

**2.2. Jack polynomials and Laplace-Beltrami operator.** Jack polynomials are a classical one-parameter deformation of Schur symmetric functions, and can be defined in several different ways. We will use a characterization via *Laplace–Beltrami operators* suggested by Stanley in his seminal paper [Sta89, note p. 85]. Since this is now a well-established theory, results of this section are given without proofs but with explicit references to the literature (mostly to Stanley’s paper [Sta89]).

First, consider the vector space  $\text{Sym}^N$  of symmetric polynomials in  $N$  variables over the field of rational functions  $\mathbb{Q}(\alpha)$ . The following differential operators act on this space:

$$D_1 = \sum_{i \leq N} \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}, \quad D_2 = \frac{1}{2} \sum_{i \leq N} x_i^2 \frac{\partial^2}{\partial x_i^2}.$$

Then the *Laplace–Beltrami operator*  $D_\alpha$  is defined as  $D_\alpha = D_1 + \alpha D_2$ .

**Proposition 2.1.** *There exists a unique family  $J_\lambda^{(\alpha)}$  (indexed by partitions  $\lambda$  of length at most  $N$ ) in  $\text{Sym}^N$  that satisfies:*

(C1)  $J_\lambda^{(\alpha)}(x_1, \dots, x_N)$  is an eigenvector of  $D_\alpha$  with eigenvalue

$$\text{ev}(\lambda) := \alpha b(\lambda) - b(\lambda^t) + (N-1)|\lambda|;$$

(C2) the monomial expansion of  $J_\lambda^{(\alpha)}$  is given by

$$(11) \quad J_\lambda = \text{hook}_\alpha(\lambda) m_\lambda + \sum_{\nu < \lambda} a_\nu^\lambda m_\nu, \text{ where } a_\nu^\lambda \in \mathbb{Q}(\alpha).$$

(Recall that we use the dominance order on partitions.)

These polynomials are called Jack polynomials.



This is not the definition of Jack polynomials used by Stanley, but the fact that Jack polynomials indeed satisfy these properties can be found in his work [Sta89, Theorem 3.1 and Theorem 5.6]. The uniqueness is an easy linear algebra exercise when one has observed that  $\text{ev}(\lambda) = \text{ev}(\mu)$  and  $|\lambda| = |\mu|$  imply that  $\lambda$  and  $\mu$  are either equal or incomparable for the dominance order; see [Sta89, Lemma 3.2]. A deep result of Knop and Sahi [KS97] asserts that  $a_\nu^\lambda$  lies in fact in  $\mathbb{N}[\alpha]$ . In particular, Jack polynomials depend polynomially on  $\alpha$ .

With the above definition, the Jack polynomial  $J_\lambda^{(\alpha)}$  depends on  $N$ , the number of variables. However, it is easy to see that it satisfies the compatibility relation  $J_\lambda^{(\alpha)}(x_1, \dots, x_N, 0) = J_\lambda^{(\alpha)}(x_1, \dots, x_N)$  and thus  $J_\lambda^{(\alpha)}$  can be seen as a symmetric function. In the sequel, when working with differential operators, we sometimes confuse a symmetric function  $f$  with its restriction  $f(x_1, \dots, x_N, 0, 0, \dots)$  to  $N$  variables.

Stanley also established the following specialization formula at  $\alpha = 0$ :

$$(12) \quad J_\lambda^{(0)} = \left( \prod_i \lambda_i^{t_i} \right) e_{\lambda^t},$$

where  $e_\lambda$  is the *elementary symmetric function* associated with  $\lambda$  [Sta89, Proposition 7.6]. A key point in his proof, that will be also important in the present paper, is the following proposition.

**Proposition 2.2.** *For any partition  $\rho \vdash n$ ,*

(1) *the elementary symmetric function  $e_\rho$  is an eigenvector of the operator  $D_1$ :*

$$D_1 e_\rho = ((N-1)|\rho| - b(\rho)) e_\rho;$$

(2) *for any partition  $\mu \vdash n$  such that  $b(\rho) = b(\mu)$  either  $\rho = \mu$  or  $\rho \not\leq \mu$ .*

Here is an easy corollary, that will be useful for us.

**Corollary 2.3.** *Fix a partition  $\lambda$  and let  $f \in \text{Sym}$  be a homogeneous symmetric function with an expansion in the monomial basis of the following form:*

$$f = \sum_{\mu < \lambda} d_\mu m_\mu,$$

*for some  $d_\mu \in \mathbb{Q}$ . If, for any number  $N$  of variables,*

$$D_1 f = ((N-1)|\lambda| - b(\lambda^t)) f$$

*then  $f = 0$ .*

*Proof.* From the first part of Proposition 2.2, the eigenspace of the operator  $D_1$  corresponding to the eigenvalue  $((N-1)|\lambda| - b(\lambda^t))$  is spanned by the functions  $e_{\rho^t}$  with  $b_{\rho^t} = b_{\lambda^t}$ . Therefore, using now the second part of Proposition 2.2, we know that the expansion of  $f$  in the elementary basis must have the following form:

$$f = c_\lambda e_{\lambda^t} + \sum_{\lambda^t \not\leq \rho^t} c_\rho e_{\rho^t}.$$

But  $\lambda^t \not\leq \rho^t$  is equivalent to  $\rho \not\leq \lambda$ . Moreover, it is easy to see that the expansion of the elementary symmetric function  $e_{\lambda^t}$  in the monomial basis involves only elements  $m_\mu$  indexed by partitions  $\mu \leq \lambda$ :

$$e_{\lambda^t} = m_\lambda + \sum_{\mu < \lambda} b_\mu^\lambda m_\mu.$$

Combining these two facts we know that the expansion of  $f$  in the monomial basis has the following form:

$$f = c_\lambda(m_\lambda + \sum_{\mu < \lambda} b_\mu^\lambda m_\mu) + \sum_{\rho \not\leq \lambda} c_\rho(m_\rho + \sum_{\mu < \rho} b_\mu^\rho m_\mu).$$

But we assumed that

$$f = \sum_{\mu < \lambda} d_\mu m_\mu,$$

which implies that  $c_\lambda = 0$  and  $c_\rho = 0$  for all  $\rho \not\leq \lambda$ , thus  $f = 0$  as claimed.  $\square$

**2.3. Goulden and Jackson's conjectures.** Following Goulden and Jackson [GJ96] we define

$$(13) \quad \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) := \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{J_\lambda^{(\alpha)}(\mathbf{x}) J_\lambda^{(\alpha)}(\mathbf{y}) J_\lambda^{(\alpha)}(\mathbf{z})}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha}.$$

$$(14) \quad \psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) := \alpha t \frac{\partial}{\partial t} \log \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha).$$

We then consider their power-sum expansion, *i.e.* the two families of coefficients  $h_{\mu,\nu}^\tau$  and  $c_{\mu,\nu}^\tau$  defined by

$$(15) \quad \psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) = \sum_{n \geq 1} t^n \sum_{\mu, \nu, \tau \vdash n} h_{\mu,\nu}^\tau (\alpha - 1) p_\tau(\mathbf{x}) p_\mu(\mathbf{y}) p_\nu(\mathbf{z});$$

$$(16) \quad \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) = \sum_{n \geq 1} t^n \sum_{\mu, \nu, \tau \vdash n} \frac{c_{\mu,\nu}^\tau (\alpha - 1)}{\alpha^{\ell(\tau)} z_\tau} p_\tau(\mathbf{x}) p_\mu(\mathbf{y}) p_\nu(\mathbf{z}).$$

The definition of the coefficients  $h_{\mu,\nu}^\tau (\alpha - 1)$  was already given in Section 1.2, we recall it here to emphasize the similarity with  $c_{\mu,\nu}^\tau (\alpha - 1)$ . Goulden and Jackson conjectured that all these coefficients are polynomials in  $\beta = \alpha - 1$  with non-negative integer coefficients and some combinatorial interpretations. The first part of their conjecture, that is the statement that  $c_{\mu,\nu}^\tau (\beta)$  are polynomials in  $\beta$  with rational coefficients was recently proven by the authors of this paper:

**Theorem 2.4.** [DF16, Proposition B.2] *For any positive integer  $n$  and for any partitions  $\mu, \nu, \tau \vdash n$ , the quantity  $c_{\mu,\nu}^\tau (\beta)$  is a polynomial in  $\beta$  (or, equivalently, in  $\alpha$ ) with rational coefficients.*

Recall from Eq. (14), that

$$\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) / \alpha = t \frac{\partial}{\partial t} \log \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha).$$

Therefore, the coefficients of the power-sum expansion of the left-hand side — that correspond to  $h_{\mu,\nu}^\tau(\beta)/\alpha$  — can be expressed as polynomials in terms of the coefficients of the power-sum expansion of  $\phi$  — that correspond to  $|\lambda|c_{\mu,\nu}^\tau(\beta)/(\alpha^{\ell(\tau)}z_\lambda)$ . In particular, an immediate corollary of the above theorem is the following one:

**Corollary 2.5.** *For any positive integer  $n$  and for any partitions  $\mu, \nu, \tau \vdash n$ , the coefficient  $h_{\mu,\nu}^\tau(\beta)$  is a rational function in  $\alpha$  with only possible pole at  $\alpha = 0$ .*

Showing that there is in fact no pole at  $\alpha = 0$ , as claimed in Theorem 1.2, requires a great deal of work and is the main contribution of this paper.

**2.4. Set partitions.** The combinatorics of set partitions is central in the theory of cumulants and will be significant in this article. We recall here some well-known facts about them.

Fix a ground set  $S$ . A *set partition* of  $S$  is a (non-ordered) family of non-empty disjoint subsets of  $S$  (called parts of the partition), whose union is  $S$ . In the following, we always assume that  $S$  is finite.

Denote by  $\mathcal{P}(S)$  the set of set partitions of  $S$ . Then  $\mathcal{P}(S)$  may be endowed with a natural partial order: the *refinement* order. We say that  $\pi$  is *finer* than  $\pi'$  (or  $\pi'$  *coarser* than  $\pi$ ) if every part of  $\pi$  is included in some part of  $\pi'$ . We denote this by  $\pi \leq \pi'$ .

Endowed with this order,  $\mathcal{P}(S)$  is a complete lattice, which means that each family  $F$  of set partitions admits a join (the finest set partition which is coarser than all set partitions in  $F$ ; we denote the join operator by  $\vee$ ) and a meet (the coarsest set partition which is finer than all set partitions in  $F$ ; we denote the meet operator by  $\wedge$ ). In particular, the lattice  $\mathcal{P}(S)$  has a maximum  $\{S\}$  (the partition in only one part) and a minimum  $\{\{x\}, x \in S\}$  (the partition in singletons).

Moreover, this lattice is ranked: the rank  $\text{rk}(\pi)$  of a set partition  $\pi$  is  $|S| - \#(\pi)$ , where  $\#(\pi)$  denotes the number of parts of  $\pi$ . The rank is compatible with the lattice structure in the following sense: for any two set partitions  $\pi$  and  $\pi'$ ,

$$(17) \quad \text{rk}(\pi \vee \pi') \leq \text{rk}(\pi) + \text{rk}(\pi').$$

Lastly, denote by  $\mu$  the Möbius function of the partition lattice  $\mathcal{P}(S)$ . In this paper, we only use evaluations of  $\mu$  at pairs  $(\pi, \{S\})$  (that is, the second argument is the one-part partition of  $S$ , which is the maximum of  $\mathcal{P}(S)$ ). In this case, the value of the Möbius function is given by:

$$(18) \quad \mu(\pi, \{S\}) = (-1)^{\#(\pi)-1} (\#(\pi) - 1)!.$$

### 3. CUMULANTS

#### 3.1. Partial cumulants.

**Definition 3.1.** Let  $(u_I)_{I \subseteq J}$  be a family of elements in a field, indexed by subsets of a finite set  $J$ . Then its *partial cumulants* are defined as follows. For any non-empty subset  $H$  of  $J$ , set

$$(19) \quad \kappa_H(\mathbf{u}) = \sum_{\pi \in \mathcal{P}(H)} \mu(\pi, \{H\}) \prod_{B \in \pi} u_B.$$

The terminology comes from probability theory. Let  $J = [r]$ , and let  $X_1, \dots, X_r$  be random variables with finite moments defined on the same probability space. Then define  $u_I = \mathbb{E}(\prod_{i \in I} X_i)$ , where  $\mathbb{E}$  denotes the expected value. The quantity  $\kappa_{[r]}(\mathbf{u})$  as defined above, is known as the joint (or mixed) cumulant of the random variables  $X_1, \dots, X_r$ . Also,  $\kappa_H(\mathbf{u})$  is the joint/mixed cumulant of the smaller family  $\{X_h, h \in H\}$ .

Joint/mixed cumulants have been studied by Leonov and Shiryaev in [LS59] (see also an older note of Schützenberger [Sch47], where they are introduced under the French name *déviations d'indépendance*). They now appear in random graph theory [JLR00, Chapter 6] and have inspired a lot of work in noncommutative probability theory; see [NS11] for a concise introduction to the topic.

Even if this probabilistic interpretation of cumulants is not relevant here, we will use several lemmas that have been discovered by the second author in a probabilistic context [Fér13].

A classical result – see, e.g., [JLR00, Proposition 6.16 (vi)] – is that relation (19) can be inverted as follows: for any non-empty subset  $H$  of  $J$ ,

$$(20) \quad u_H = \sum_{\pi \in \mathcal{P}(H)} \prod_{B \in \pi} \kappa_B(\mathbf{u}).$$

**3.2. A multiplicative criterion for small cumulants.** Let  $R$  be a field and  $\alpha$  a formal parameter. Denote by  $R(\alpha)$  the field of rational functions in  $\alpha$  with coefficients in  $R$ . In all applications in this paper,  $\alpha$  is the Jack parameter.

*Definition 3.2.* We use the following notation: for  $r \in R(\alpha)$  and an integer  $k$ , we write  $r = O(\alpha^k)$  if the rational function  $r \cdot \alpha^{-k}$  has no pole in 0.

As above, we consider a family  $\mathbf{u} = (u_I)_{I \subseteq [r]}$  of elements of  $R(\alpha)$  indexed by subsets of  $[r]$ . Throughout this section, we also assume that these elements are **non-zero** and  $u_\emptyset = 1$ .

In addition to partial cumulants, we also define the *cumulative factorization error terms*  $T_H(\mathbf{u})$  of the family  $\mathbf{u}$ . The quantities  $T_H(\mathbf{u})_{H \subseteq [r], |H| \geq 2}$  are inductively defined as follows: for any subset  $G$  of  $[r]$  of size at least 2,

$$(21) \quad u_G = \prod_{g \in G} u_{\{g\}} \cdot \prod_{\substack{H \subseteq G \\ |H| \geq 2}} (1 + T_H(\mathbf{u})).$$

Using inclusion-exclusion principle, a direct equivalent definition is the following one: for any subset  $H$  of  $[r]$  of size at least 2, set

$$(22) \quad T_H(\mathbf{u}) = \prod_{G \subseteq H} u_G^{(-1)^{|H|}} - 1.$$

We have the following result.

**Proposition 3.3.** *Using the above notation, the following statements are equivalent:*

I. Strong factorization property: *for any subset  $H \subseteq [r]$  of size at least 2, one has*

$$(23) \quad T_H(\mathbf{u}) = O\left(\alpha^{|H|-1}\right).$$

II. Small cumulant property: *for any subset  $H \subseteq [r]$  of size at least 2, one has*

$$(24) \quad \kappa_H(\mathbf{u}) = \left(\prod_{h \in H} u_h\right) O\left(\alpha^{|H|-1}\right).$$

This proposition is a reformulation of [Fér13, Lemma 2.2]. However, the context and notation are quite different: in [Fér13], the author is interested in sequences of random variables, while here we consider rational functions in  $\alpha$ . Thus, we prefer to copy the proof here, adapting it to our context.

*Proof.* We first assume that  $u_{\{i\}} = 1$  for all  $i$  in  $[r]$ .

Let us first show that I implies II. Assume that  $T_H(\mathbf{u}) = O\left(\alpha^{|H|-1}\right)$ , for any  $H \subseteq [r]$  of size at least 2. The goal is to prove that  $\kappa_{[r]}(\mathbf{u}) = O\left(\alpha^{r-1}\right)$ . This corresponds only to the case  $H = [r]$  of II, but the same proof will work for any  $H \subseteq [r]$ .

Fix a set partition  $\pi \in \mathcal{P}(r)$ . For each block  $B$  of  $\pi$ , we expand the second product in Eq. (21):

$$u_B = \sum_{\{H_1, \dots, H_m\} \subseteq 2_{\geq 2}^B} T_{H_1}(\mathbf{u}) \cdots T_{H_m}(\mathbf{u}),$$

where  $2_{\geq 2}^B$  denotes the set of subsets of  $B$  of size at least 2; the sum here runs over all (unordered) subsets of  $2_{\geq 2}^B$  (in particular, the size  $m$  of this subset is not fixed). Therefore,

$$\prod_{B \in \pi} u_B = \sum_{\{H_1, \dots, H_m\} \subseteq 2_{\geq 2}^{[r]}, (\star)} T_{H_1}(\mathbf{u}) \cdots T_{H_m}(\mathbf{u}),$$

where the sum runs over all sets of subsets of  $[r]$  of size at least 2 with the following property, denoted by  $(\star)$ : each  $H_i$  is contained in some block of  $\pi$ . In other terms, for each  $i \in [m]$ ,  $\pi$  must be coarser than the partition  $\Pi(H_i)$ , which, by definition, has  $H_i$  and singletons as blocks. Using Eq. (19) and reorganizing, we get

$$(25) \quad \kappa_{[r]}(\mathbf{u}) = \sum_{\{H_1, \dots, H_m\} \subseteq 2_{\geq 2}^{[r]}} T_{H_1}(\mathbf{u}) \cdots T_{H_m}(\mathbf{u}) \left( \sum_{\substack{\pi \in \mathcal{P}([r]) \\ \forall i, \pi \geq \Pi(H_i)}} \mu(\pi, \{[r]\}) \right).$$

The condition on  $\pi$  can be rewritten as

$$\pi \geq \Pi(H_1) \vee \cdots \vee \Pi(H_m).$$

Hence, by definition of the Möbius function, the sum in the parenthesis is equal to 0, unless  $\Pi(H_1) \vee \cdots \vee \Pi(H_m) = \{[r]\}$  (in other terms, unless the hypergraph with

vertex set  $[r]$  and edges  $(H_i)_{1 \leq i \leq m}$  is connected). On the one hand, by Eq. (17), it may happen only if:

$$\sum_{i=1}^m \text{rk}(\Pi(H_i)) = \sum_{i=1}^m (|H_i| - 1) \geq \text{rk}([r]) = r - 1.$$

On the other hand, one has

$$T_{H_1}(\mathbf{u}) \cdots T_{H_m}(\mathbf{u}) = O\left(\alpha^{\sum_{i=1}^m (|H_i| - 1)}\right).$$

Hence only summands of order of magnitude  $O(\alpha^k)$  for  $k \geq r - 1$  survive and one has

$$\kappa_{[r]}(\mathbf{u}) = O(\alpha^{r-1}),$$

as wanted.

Let us now consider the converse statement. We proceed by induction on  $r$  and we assume that for all  $r'$  smaller than a given  $r \geq 2$  the proposition holds.

Consider some family  $(u_I)_{I \subseteq [r]}$  such that II holds. By induction hypothesis, for all  $H \subsetneq [r]$ , one has  $T_H(\mathbf{u}) = O(\alpha^{|H|-1})$ . Note that Eq. (21) then implies  $u_H = O(1)$  and  $u_H^{-1} = O(1)$  for  $H \subsetneq [r]$ . It remains to prove that

$$T_{[r]}(\mathbf{u}) = \prod_{H \subseteq [r]} (u_H)^{(-1)^{|H|}} - 1 = O(\alpha^{r-1}).$$

Thanks to the estimates above for  $u_H$ , this can be rewritten as

$$(26) \quad u_{[r]} = \prod_{H \subsetneq [r]} (u_H)^{(-1)^{r-1-|H|}} + O(\alpha^{r-1}).$$

Define now an auxiliary family  $\mathbf{v}$ :

$$v_G = \begin{cases} u_G & \text{if } G \subsetneq [r]; \\ \prod_{H \subsetneq [r]} (u_H)^{(-1)^{r-1-|H|}} & \text{for } G = [r]. \end{cases}$$

Clearly, since  $T_G(\mathbf{v}) = T_G(\mathbf{u})$  for  $G \subsetneq [r]$  and  $T_{[r]}(\mathbf{v}) = 0$ , the family  $\mathbf{v}$  has the strong factorization property. Thus, using the first part of the proof, it also has the small cumulant property. In particular:

$$\kappa_{[r]}(\mathbf{v}) = O(\alpha^{r-1}).$$

But, by hypothesis,

$$\kappa_{[r]}(\mathbf{u}) = O(\alpha^{r-1}).$$

As  $v_H = u_H$  for  $H \subsetneq [r]$ , one has:

$$u_{[r]} - v_{[r]} = \kappa_{[r]}(\mathbf{u}) - \kappa_{[r]}(\mathbf{v}) = O(\alpha^{r-1}),$$

which proves Eq. (26).

The general case follows directly from the case  $u_{\{i\}} = 1$  by considering the family  $u'_I = u_I / \prod_{i \in I} u_{\{i\}}$ . Indeed, for  $|H| \geq 2$ , it holds that

$$\begin{aligned} T_H(\mathbf{u}') &= T_H(\mathbf{u}); \\ K_H(\mathbf{u}') &= K_H(\mathbf{u}) \cdot \left( \prod_{h \in H} u_{\{h\}} \right)^{-1}. \end{aligned} \quad \square$$

A first consequence of this multiplicative criterion for small cumulants is the following stability result.

**Corollary 3.4.** *Consider two families  $(u_I)_{I \subseteq [r]}$  and  $(v_I)_{I \subseteq [r]}$  with the small cumulant property. Then their entry-wise product  $(u_I v_I)_{I \subseteq [r]}$  and quotient  $(u_I / v_I)_{I \subseteq [r]}$  also have the small cumulant property.*

*Proof.* This is trivial for the strong factorization property and the small cumulant property is equivalent to it.  $\square$

**3.3. Hook cumulants.** To illustrate the above propositions and as a preparation for our next results, we show in this section that some families involving hook polynomials have the small cumulant property.

We first consider  $\text{hook}_\alpha$  and start with a technical lemma.

**Lemma 3.5.** *Fix a positive integer  $r$  and a subset  $K$  of  $[r]$ . Let  $(c_i)_{i \in K}$  be a family of elements of  $R(\alpha)$ , and let  $C \in R(\alpha)$  with  $C \neq 0$ . Assume that  $C$ ,  $C^{-1}$  and all  $c_i$  are  $O(1)$ . For a subset  $I$  of  $K$  we define*

$$v_I = C + \alpha \cdot \sum_{i \in I} c_i.$$

*Then we have, for any subset  $H$  of  $K$ ,*

$$T_H(\mathbf{v}) = O(\alpha^{|H|}).$$

This is a reformulation of [Fér13, Lemma 2.4], but, again, as notation is quite different there, we adapt the proof to our context.

*Proof.* It is enough to prove the statement for  $H = K$ . Indeed, the case of a general set  $H$  follows by considering the same family restricted to subsets of  $H$ .

Define  $R_{\text{even}}$  (resp.  $R_{\text{odd}}$ ) as

$$\prod_{\delta} \left( C + \alpha \sum_{i \in \delta} c_i \right),$$

where the product runs over subsets of  $K$  of even (resp. odd) size. With this notation,  $T_K(\mathbf{v}) = R_{\text{even}}/R_{\text{odd}} - 1 = (R_{\text{even}} - R_{\text{odd}})/R_{\text{odd}}$ . Since  $R_{\text{odd}}^{-1} = O(1)$  (each term in the product is  $O(1)$ , as well as its inverse), it is enough to show that  $R_{\text{even}} - R_{\text{odd}} = O(\alpha^{|K|})$ .

Expanding the product in the definition of  $R_{\text{even}}$ , one gets

$$R_{\text{even}} = \sum_{m \geq 0} \frac{1}{m!} \sum_{\delta_1, \dots, \delta_m} \sum_{i_1 \in \delta_1, \dots, i_m \in \delta_m} \alpha^m c_{i_1} \cdots c_{i_m} C^{2^{|K|-1}-m}.$$

The index set of the second summation symbol is the set of lists of  $m$  distinct (but not necessarily disjoint) subsets of  $K$  of even size. Of course, a similar formula with subsets of odd size holds for  $R_{\text{odd}}$ .

Let us fix an integer  $m < |K|$  and a list  $i_1, \dots, i_m$ . Denote by  $i_0$  the smallest integer in  $K$  different from  $i_1, \dots, i_m$  (as  $m < |K|$ , such an integer necessarily exists). Then one has a bijection:

$$\left\{ \begin{array}{l} \text{lists of subsets} \\ \delta_1, \dots, \delta_m \text{ of even size such} \\ \text{that, for all } h \leq m, i_h \in \delta_h \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{lists of subsets} \\ \delta_1, \dots, \delta_m \text{ of odd size such} \\ \text{that, for all } h \leq m, i_h \in \delta_h \end{array} \right\}$$

$$(\delta_1, \dots, \delta_m) \mapsto (\delta_1 \nabla \{i_0\}, \dots, \delta_m \nabla \{i_0\}),$$

where  $\nabla$  is the symmetric difference operator. This bijection implies that the summand  $\alpha^m c_{i_1} \cdots c_{i_m} C^{2^{|K|-2}-m}$  appears as many times in  $R_{\text{even}}$  as in  $R_{\text{odd}}$ . Finally, in the difference  $R_{\text{even}} - R_{\text{odd}}$ , terms corresponding to values of  $m$  smaller than  $|K|$  cancel each other and one has

$$R_{\text{even}} - R_{\text{odd}} = O(\alpha^{|K|}). \quad \square$$

We recall that given partitions  $\lambda^1, \dots, \lambda^r$  and a subset  $I$  of  $[r]$  we set

$$\lambda^I := \bigoplus_{i \in I} \lambda^i,$$

where  $\oplus$  is the entrywise sum; see Section 2.1.

**Proposition 3.6.** *Fix some partitions  $\lambda^1, \dots, \lambda^r$  and for a subset  $I$  of  $[r]$  denote  $u_I = \text{hook}_\alpha(\lambda^I)$ . The family  $(u_I)$  has the strong factorization property, and hence, the small cumulant property.*

*Proof.* It is enough to prove that  $T_{[r]}(\mathbf{u}) = O(\alpha^{r-1})$ .

Fix some subset  $I = \{i_1, \dots, i_t\}$  of  $[r]$  with  $i_1 < \dots < i_t$ . Observe that the Young diagram  $\lambda^I$  can be constructed by sorting the columns of the diagrams  $\lambda^{i_1}, \dots, \lambda^{i_t}$  in decreasing order. When several columns have the same length, we put first the columns of  $\lambda^{i_1}$ , then those of  $\lambda^{i_2}$  and so on; see Fig. 2 (at the moment, please disregard symbols in boxes). This gives a way to identify boxes of  $\lambda^I$  with boxes of the diagrams  $\lambda^{i_s}$  ( $1 \leq s \leq t$ ) that we shall use below.

With this identification, if  $b = (c, r)$  is a box in  $\lambda^g$  for some  $g \in I$ , its leg-length in  $\lambda^I$  is the same as in  $\lambda^g$ . We denote it by  $\ell(b)$ .

However, the arm-length of  $b$  in  $\lambda^I$  may be bigger than the one in  $\lambda^g$ . We denote these two quantities by  $a_I(b)$  and  $a_g(b)$ . Let us also define  $a_i(b)$  for  $i \neq g$  in  $I$ , as follows:

- for  $i < g$ ,  $a_i(b)$  is the number of boxes  $b'$  in the  $r$ -th row of  $\lambda^i$  such that the size of the column of  $b'$  is **smaller than** the size of the column of  $b$  (e.g., on Fig. 2, for  $i = 1$ , these are boxes with a diamond);



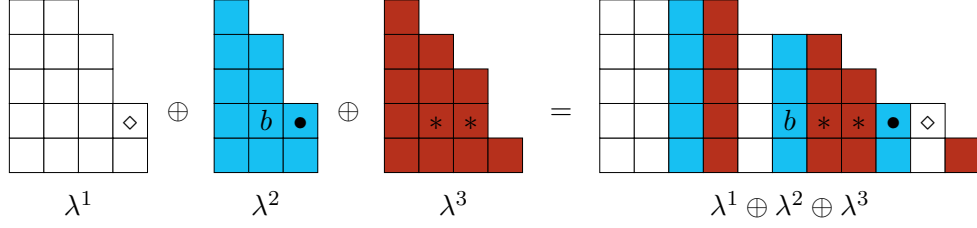


Figure 2. The diagram of an entry-wise sum of partitions.

- for  $i > g$ ,  $a_i(b)$  is the number of boxes  $b'$  in the  $r$ -th row of  $\lambda^i$  such that the size of the column of  $b'$  is **at most** the size of the column of  $b$  (e.g., on Fig. 2, for  $i = 3$ , these are boxes with an asterisk).

Looking at Fig. 2, it is easy to see that

$$(27) \quad a_I(b) = \sum_{i \in I} a_i(b).$$

Therefore, for  $G \subseteq [r]$ , one has:

$$u_G = \text{hook}_\alpha \left( \bigoplus_{g \in G} \lambda^g \right) = \prod_{g \in G} \left[ \prod_{b \in \lambda^g} \ell(b) + 1 + \alpha a_G(b) \right].$$

From the definition of  $T_{[r]}(\mathbf{u})$ , given by Eq. (22), we get:

$$(28) \quad 1 + T_{[r]}(\mathbf{u}) = \prod_{G \subseteq [r]} \prod_{g \in G} \left[ \prod_{b \in \lambda^g} \ell(b) + 1 + \alpha a_G(b) \right]^{(-1)^{r-|G|}} \\ = \prod_{g \in [r]} \prod_{b \in \lambda^g} \left[ \prod_{\substack{G \subseteq [r] \\ G \ni g}} \left( \ell(b) + 1 + \alpha a_G(b) \right)^{(-1)^{r-|G|}} \right].$$

The expression inside the square bracket corresponds to  $1 + T_{[r] \setminus \{g\}}(\mathbf{v}^b)$ , where  $(\mathbf{v}^b)$  is a family indexed by subsets of  $[r] \setminus \{g\}$  defined as follows: for a subset  $I$  of  $[r] \setminus \{g\}$  we set

$$v_I^b := \ell(b) + 1 + \alpha a_{I \cup \{g\}}(b).$$

From Eq. (27), we observe that  $v_I^b$  is as in Lemma 3.5 with the following values of the parameters:  $K = [r] \setminus \{g\}$ ,  $C = \ell(b) + 1 + \alpha a_g(b)$ , and  $c_i = a_i(b)$  for  $i \neq g$ . Therefore we conclude that

$$T_{[r] \setminus \{g\}}(\mathbf{v}^b) = O(\alpha^{r-1}).$$

Going back to Eq. (28), we have:

$$1 + T_{[r]}(\mathbf{u}) = \prod_{g \in [r]} \prod_{b \in \lambda^g} \left( 1 + T_{[r] \setminus \{g\}}(\mathbf{v}^b) \right) = 1 + O(\alpha^{r-1}),$$

which completes the proof.  $\square$

Let us now look at the second hook-polynomial  $\text{hook}'_\alpha$ . If we try to follow the same argument as above, we want to apply Lemma 3.5 with  $K = [r] \setminus \{g\}$ ,  $C = \ell(b) + \alpha(1 + a_g(b))$ , and  $c_i = a_i(b)$  for  $i \neq g$ . Note, however, that if the box  $b$  has leg-length 0, then  $C = 0$  for  $\alpha = 0$ , and in this case the hypothesis  $C^{-1} = O(1)$  of Lemma 3.5 is not fulfilled. To overcome this difficulty, we define

$$\text{hook}''_\alpha(\lambda) = \prod_{\substack{\square \in \lambda \\ \ell(\square) \neq 0}} (\alpha a(\square) + \ell(\square) + \alpha).$$

By definition, the top-most box of each column of a diagram  $\lambda$  has leg-length 0. Moreover  $\lambda$  has  $m_i(\lambda^t)$  columns of height  $i$ , thus the arm-length of the top-most boxes of these columns are  $0, 1, \dots, m_i(\lambda^t) - 1$  respectively. Therefore,

$$\prod_{\substack{\square \in \lambda \\ \ell(\square) = 0}} (\alpha a(\square) + \ell(\square) + \alpha) = \alpha^{\lambda_1} \prod_i m_i(\lambda^t)!,$$

so that

$$(29) \quad \text{hook}'_\alpha(\lambda) = \alpha^{\lambda_1} \left( \prod_i m_i(\lambda^t)! \right) \text{hook}''_\alpha(\lambda).$$

Now, the exact same proof as for  $\text{hook}_\alpha$  yields the following result:

**Proposition 3.7.** *Fix some partitions  $\lambda^1, \dots, \lambda^r$  and set  $v_I = \text{hook}''_\alpha(\lambda^I)$  for all subsets  $I$  of  $[r]$ . The family  $(v_I)$  has the strong factorization property, and hence, the small cumulant property.*

#### 4. STRONG FACTORIZATION PROPERTY OF JACK POLYNOMIALS

Let us fix partitions  $\lambda^1, \dots, \lambda^r$ , and for any subset  $I \subseteq [r]$  we define  $u_I := J_{\lambda^I}$ . The purpose of this section is to prove Theorem 1.4; with the notations above, it writes as  $\kappa^J(\lambda^1, \dots, \lambda^r) = \kappa_{[r]}(\mathbf{u}) = O(\alpha^{r-1})$ . We start with some preliminary results.

##### 4.1. Preliminary results.

**Proposition 4.1.** *For any partitions  $\lambda^1, \dots, \lambda^r$  the cumulant of Jack polynomials has a monomial expansion of the following form:*

$$\kappa^J(\lambda^1, \dots, \lambda^r) = \sum_{\mu < \lambda^{[r]}} c_\mu^{\lambda^1, \dots, \lambda^r} m_\mu + O(\alpha^{r-1}),$$

where the coefficients  $c_\mu^{\lambda^1, \dots, \lambda^r}$  are polynomials in  $\alpha$ .

*Proof.* First, observe that for any partitions  $\nu^1$  and  $\nu^2$ , one has

$$m_{\nu^1} m_{\nu^2} = m_{\nu^1 \oplus \nu^2} + \sum_{\mu < \nu^1 \oplus \nu^2} b_\mu^{\nu^1, \nu^2} m_\mu,$$

for some integers  $b_\mu^{\nu^1, \nu^2}$ .

Fix partitions  $\lambda^1, \dots, \lambda^r$  and a set partition  $\pi = \{\pi_1, \dots, \pi_s\} \in \mathcal{P}([r])$ . Note that  $\lambda^{\pi_1} \oplus \dots \oplus \lambda^{\pi_s} = \lambda^{[r]}$ . Thanks to Eq. (11) and the above observation on products of monomials, there exist coefficients  $d_\mu^{\lambda^{\pi_1}, \dots, \lambda^{\pi_s}} \in \mathbb{Q}[\alpha]$  such that:

$$J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} = \text{hook}_\alpha(\lambda^{\pi_1}) \cdots \text{hook}_\alpha(\lambda^{\pi_s}) m_{\lambda^{[r]}} + \sum_{\mu < \lambda^{[r]}} d_\mu^{\lambda^{\pi_1}, \dots, \lambda^{\pi_s}} m_\mu.$$

As a consequence, there exist coefficients  $c_\mu^{\lambda^1, \dots, \lambda^r} \in \mathbb{Q}[\alpha]$  such that

$$\kappa^J(\lambda^1, \dots, \lambda^r) = \kappa_{[r]}(v) m_{\lambda^{[r]}} + \sum_{\mu < \lambda^{[r]}} c_\mu^{\lambda^1, \dots, \lambda^r} m_\mu,$$

where  $v_I = \text{hook}_\alpha(\lambda^I)$ . Proposition 3.6 completes the proof.  $\square$

For any positive integer  $r$  and for any partitions  $\lambda^1, \dots, \lambda^r$  we define

$$(30) \quad \text{InEx}(\lambda^1, \dots, \lambda^r) := \sum_{I \subseteq [r]} (-1)^{r-|I|} b(\lambda^I),$$

where  $b(\lambda) = \sum_i \binom{\lambda_i}{2}$ , as defined in (10).

**Proposition 4.2.** *Let  $r \geq 3$  be a positive integer. Then, for any partitions  $\lambda^1, \dots, \lambda^r$  one has:*

$$\text{InEx}(\lambda^1, \dots, \lambda^r) = 0.$$

*Proof.* Expanding the definition and completing partitions with zeros, we have:

$$\text{InEx}(\lambda^1, \dots, \lambda^r) = \sum_{j \geq 1} \sum_{I \subseteq [r]} (-1)^{r-|I|} \binom{\lambda_j^I}{2}.$$

In particular, it is enough to prove that the summand corresponding to any given  $j \geq 1$  is equal to 0. In other terms, we can restrict ourselves to the case where  $\lambda^i = (\lambda_1^i)$  has only one part.

In this case,  $\text{InEx}(\lambda^1, \dots, \lambda^r)$  is a symmetric polynomial in  $\lambda_1^1, \dots, \lambda_1^r$  of degree 2 without constant term. Moreover, its coefficients are given by:

$$-[\lambda_1^1] \text{InEx}(\lambda^1, \dots, \lambda^r) = [(\lambda_1^1)^2] \text{InEx}(\lambda^1, \dots, \lambda^r) = \sum_{[1] \subseteq I \subseteq [r]} \frac{(-1)^{r-|I|}}{2} = 0,$$

and

$$[\lambda_1^1 \cdot \lambda_1^2] \text{InEx}(\lambda^1, \dots, \lambda^r) = \sum_{[2] \subseteq I \subseteq [r]} (-1)^{r-|I|} = 0.$$

This completes the proof.  $\square$

Let us now define two functions that will be of great importance in the proof of Theorem 1.4:

$$(31) \quad A_1(\lambda^1, \dots, \lambda^r) := \sum_{\pi \in \mathcal{P}([r])} \left( \mu(\pi, \{[r]\}) \left( \sum_{B \in \pi} b(\lambda^B) \right) \prod_{B \in \pi} J_{\lambda^B} \right),$$

$$(32) \quad A_2(\lambda^1, \dots, \lambda^r) := \sum_{\pi \in \mathcal{P}([r])} \mu(\pi, \{[r]\}) D_{1,2}(J_{\lambda^B} : B \in \pi),$$

where  $D_{1,2}$  is a multivariate operator defined as follows:  $D_{1,2}(f_1) = 0$  and, for  $k \geq 2$ ,

$$(33) \quad D_{1,2}(f_1, \dots, f_k) := \sum_{1 \leq m \leq N} \sum_{1 \leq i < j \leq k} f_1 \cdots \left( x_m \frac{\partial}{\partial x_m} f_i \right) \cdots \left( x_m \frac{\partial}{\partial x_m} f_j \right) \cdots f_k.$$

We also recall that for any subset  $I \subseteq [r]$  we defined  $u_I := J_{\lambda^I}$ , thus

$$\kappa_I(\mathbf{u}) = \kappa^J(\lambda^i : i \in I).$$

**Lemma 4.3.** *For any positive integer  $r \geq 2$  and any partitions  $\lambda^1, \dots, \lambda^r$ , the following equality holds true:*

$$A_1(\lambda^1, \dots, \lambda^r) = b(\lambda^{[r]}) \kappa_{[r]}(\mathbf{u}) + \frac{1}{2} \sum_{\emptyset \subsetneq I \subsetneq [r]} \text{InEx}(\lambda^I, \lambda^{I^c}) \kappa_I(\mathbf{u}) \kappa_{I^c}(\mathbf{u}),$$

where  $I^c = [r] \setminus I$ .

*Proof.* We start by the following easy identity, following from Eq. (20):

$$\prod_{B \in \pi} J_{\lambda^B} = \sum_{\substack{\sigma \in \mathcal{P}([r]) \\ \sigma \leq \pi}} \left( \prod_{C \in \sigma} \kappa_C(\mathbf{u}) \right).$$

Substituting this into the definition of  $A_1$  — Eq. (31) — we obtain

$$A_1(\lambda^1, \dots, \lambda^r) = \sum_{\sigma \in \mathcal{P}([r])} \left( \sum_{\pi \in \mathcal{P}([r]); \pi \geq \sigma} \mu(\pi, \{[r]\}) \left( \sum_{B \in \pi} b(\lambda^B) \right) \right) \prod_{C \in \sigma} \kappa_C(\mathbf{u}).$$

Fix a set partition  $\sigma \in \mathcal{P}([r])$ . We claim that the value of the biggest bracket is equal to

$$(34) \quad \sum_{\pi \in \mathcal{P}([r]); \pi \geq \sigma} \mu(\pi, \{[r]\}) \left( \sum_{B \in \pi} b(\lambda^B) \right) = \text{InEx}(\lambda^B : B \in \sigma).$$

Indeed, let us order the blocks of  $\sigma$  in some way:  $\sigma = \{B_1, \dots, B_{\#(\sigma)}\}$ . Partitions  $\pi$  coarser than  $\sigma$  are in bijection with partitions of the blocks of  $\sigma$ , that is partitions of  $[\#(\sigma)]$ . Therefore the left-hand side of (34) can be rewritten as:

$$\sum_{\pi \in \mathcal{P}([r]); \pi \geq \sigma} \mu(\pi, \{[r]\}) \left( \sum_{B \in \pi} b(\lambda^B) \right) = \sum_{\rho \in \mathcal{P}([\#(\sigma)])} \mu(\rho, \{[\#(\sigma)]\}) \left( \sum_{C \in \rho} b \left( \bigoplus_{j \in C} \lambda^{B_j} \right) \right).$$

Fix some subset  $C$  of  $[\#(\sigma)]$ . The coefficient of  $b \left( \bigoplus_{j \in C} \lambda^{B_j} \right)$  in the above sum is equal to

$$a_C := \sum_{\substack{\rho \in \mathcal{P}([\#(\sigma)]) \\ C \in \rho}} \mu(\rho, \{[\#(\sigma)]\}).$$

Set-partitions  $\rho$  of  $[\#(\sigma)]$  that have  $C$  as a block are uniquely expressed as  $\{C\} \cup \rho'$ , where  $\rho'$  is a set partition of  $[\#(\sigma)] \setminus C$ . Thus

$$\begin{aligned} a_C &= \sum_{\rho' \in \mathcal{P}([\#(\sigma)] \setminus C)} \mu(\{C\} \cup \rho', \{[\#(\sigma)]\}) \\ &= \sum_{0 \leq i \leq \#(\sigma) - |C|} S(\#(\sigma) - |C|, i) i! (-1)^i = (-1)^{\#(\sigma) - |C|}, \end{aligned}$$

where  $S(n, k)$  is the *Stirling number of the second kind* and the last equality comes from the relation

$$\sum_{0 \leq k \leq n} S(n, k) (x)_k = x^n$$

evaluated at  $x = -1$ ; here,  $(x)_k := x(x-1) \cdots (x-k+1)$  denotes the falling factorial. This finishes the proof of Eq. (34).

This also completes the proof of the lemma by noticing that the right hand side of Eq. (34) vanishes for all set partitions  $\sigma$  such that  $\#(\sigma) \geq 3$ , which is ensured by Proposition 4.2.  $\square$

**Lemma 4.4.** *For any positive integer  $r \geq 2$  and any partitions  $\lambda^1, \dots, \lambda^r$ , the following equality holds true:*

$$(35) \quad A_2(\lambda^1, \dots, \lambda^r) = -\frac{1}{2} \sum_{1 \leq m \leq N} \sum_{\emptyset \subsetneq I \subsetneq [r]} \left( x_m \frac{\partial}{\partial x_m} \kappa_I(\mathbf{u}) \right) \left( x_m \frac{\partial}{\partial x_m} \kappa_{I^c}(\mathbf{u}) \right).$$

*Proof.* Let us call RHS the right-hand side of Eq. (35). Using the definition of cumulants and Leibniz rule for the operator  $x_m \frac{\partial}{\partial x_m}$  we get

$$-2 \text{ RHS} = \sum_{\substack{\emptyset \subsetneq I \subsetneq [r] \\ \pi^1 \in \mathcal{P}(I), \pi^2 \in \mathcal{P}(I^c)}} \mu(\pi^1, \{I\}) \mu(\pi^2, \{I^c\}) \left( \sum_{\substack{B^1 \in \pi^1 \\ B^2 \in \pi^2}} V_{B^1, B^2; C_1, \dots, C_s} \right),$$

where  $C_1, \dots, C_s$  are the blocks of  $\pi^1$  and  $\pi^2$  distinct from  $B^1$  and  $B^2$  and

$$V_{B^1, B^2; C_1, \dots, C_s} = \sum_{1 \leq m \leq N} \left( x_m \frac{\partial}{\partial x_m} u_{B^1} \right) \left( x_m \frac{\partial}{\partial x_m} u_{B^2} \right) \left( \prod_{i=1}^s u_{C_i} \right).$$

Fix some partition  $\{B^1, B^2, C_1, \dots, C_s\}$  with two marked blocks  $B^1$  and  $B^2$  (the order of two marked blocks matters) and consider the coefficient of  $V_{B^1, B^2; C_1, \dots, C_s}$  in  $-2 \text{ RHS}$ . Pairs of set partitions  $(\pi^1, \pi^2)$  contributing to this coefficient are obtained as follows: take a subset  $J$  of  $[s]$  and set

$$\pi^1 = B^1 \cup \{C_j, j \in J\}, \quad \pi^2 = B^2 \cup \{C_j, j \in [s] \setminus J\}.$$

Then  $\mu(\pi^1, \{I\}) = (-1)^{|J|}(|J|)!$  and  $\mu(\pi^2, \{I^c\}) = (-1)^{s-|J|}(s - |J|)!$ . Thus the coefficient of  $V_{B^1, B^2, C_1, \dots, C_s}$  in  $-2$  RHS is equal to

$$\begin{aligned} \sum_{J \subset [s]} (-1)^{|J|}(|J|)!(-1)^{s-|J|}(s - |J|)! &= \sum_{k=0}^s \binom{s}{k} (-1)^k k! (-1)^{s-k} (s - k)! \\ &= \sum_{k=0}^s (-1)^s s! = (-1)^s (s + 1)!. \end{aligned}$$

Finally, we get

$$\text{RHS} = \frac{1}{2} \sum_{\{B^1, B^2; C_1, \dots, C_s\}} (-1)^{s+1} (s + 1)! V_{B^1, B^2; C_1, \dots, C_s},$$

where the sum runs over set partitions  $\{B^1, B^2; C_1, \dots, C_s\}$  with two *ordered* marked blocks. Note that  $(-1)^{s+1} (s + 1)!$  is simply the Möbius function of the underlying set partition (forgetting the marked blocks) and that one can remove the factor  $1/2$  by summing over set partitions with two *unordered* marked blocks.

On the other hand, from the definition of  $D_{1,2}$  — Eq. (33) — for any set partition  $\pi$ , one has:

$$D_{1,2}(u_B; B \in \pi) = \sum_{\dots} V_{B^1, B^2; C_1, \dots, C_s},$$

where the sum runs over all ways to mark (in an unordered way) two blocks of  $\pi$ ; the resulting marked partition is then denoted  $\{B^1, B^2; C_1, \dots, C_s\}$  as usual. Therefore, one has

$$\sum_{\pi \in \mathcal{P}([r])} \mu(\pi, \{[r]\}) D_{1,2}(u_B; B \in \pi) = \text{RHS},$$

as claimed in the lemma.  $\square$

#### 4.2. Proof of Theorem 1.4.

*Proof of Theorem 1.4.* The proof will be given by induction on  $r$ . For  $r = 1$ , we want to prove that the Jack polynomial  $J_\lambda^{(\alpha)}$  has no singularity in  $\alpha = 0$ . This follows, *e.g.*, from the specialization for  $\alpha = 0$  given in Eq. (12). Moreover, we observed before stating Theorem 1.4 that the case  $r = 2$  also follows from Eq. (12).

Let us assume that the statement holds true for all  $m < r$ . Notice first that, by Leibniz rule, for any  $f_1, \dots, f_k \in \text{Sym}$ , one has the following expansions:

$$\begin{aligned} D_1(f_1 \cdots f_k) &= \sum_{1 \leq i \leq k} f_1 \cdots (D_1 f_i) \cdots f_k; \\ D_2(f_1 \cdots f_k) &= \sum_{1 \leq i \leq k} f_1 \cdots (D_2 f_i) \cdots f_k \\ &\quad + D_{1,2}(f_1, \dots, f_k), \end{aligned}$$

where  $D_{1,2}$  is given by Eq. (33).

Fix some partitions  $\lambda^1, \dots, \lambda^r$  and a set partition  $\pi$  of  $[r]$ . Then, one has

$$\begin{aligned}
D_\alpha (J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}}) &= \sum_{1 \leq i \leq s} J_{\lambda^{\pi_1}} \cdots (D_\alpha J_{\lambda^{\pi_i}}) \cdots J_{\lambda^{\pi_s}} + \alpha D_{1,2} (J_{\lambda^{\pi_1}}, \dots, J_{\lambda^{\pi_s}}) \\
&= \left( \sum_{1 \leq i \leq s} \left( (N-1)|\lambda^{\pi_i}| - b((\lambda^{\pi_i})^t) \right) \right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} \\
&\quad + \alpha \left( \left( \sum_{1 \leq i \leq s} b(\lambda^{\pi_i}) \right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} + D_{1,2} (J_{\lambda^{\pi_1}}, \dots, J_{\lambda^{\pi_s}}) \right) \\
&= \left( (N-1)|\lambda^{[r]}| - b((\lambda^{[r]})^t) \right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} \\
&\quad + \alpha \left( \left( \sum_{1 \leq i \leq s} b(\lambda^{\pi_i}) \right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} + D_{1,2} (J_{\lambda^{\pi_1}}, \dots, J_{\lambda^{\pi_s}}) \right),
\end{aligned}$$

where the second equality comes from Proposition 2.2. Multiplying by the appropriate value of the Möbius function and summing over set partitions  $\pi$ , it gives us the following identity:

$$\begin{aligned}
(36) \quad D_\alpha \kappa_{[r]}(\mathbf{u}) &= \left( (N-1)|\lambda^{[r]}| - b((\lambda^{[r]})^t) \right) \kappa_{[r]}(\mathbf{u}) \\
&\quad + \alpha (A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)),
\end{aligned}$$

where  $A_1$  and  $A_2$  are given by Eq. (31) and Eq. (32), respectively.

Consider the coefficient of  $\alpha^j$  in the above expression. We have

$$\begin{aligned}
[\alpha^j] D_\alpha \kappa_{[r]}(\mathbf{u}) &= \left( (N-1)|\lambda^{[r]}| - b((\lambda^{[r]})^t) \right) [\alpha^j] \kappa_{[r]}(\mathbf{u}) \\
&\quad + [\alpha^{j-1}] (A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)).
\end{aligned}$$

On the other hand, since  $D_\alpha = D_1 + \alpha D_2$ , one has

$$(37) \quad [\alpha^j] D_\alpha \kappa_{[r]}(\mathbf{u}) = D_1 ([\alpha^j] \kappa_{[r]}(\mathbf{u})) + D_2 ([\alpha^{j-1}] \kappa_{[r]}(\mathbf{u})).$$

Comparing both expressions, we have the following identity, which will be a key tool in the proof:

$$\begin{aligned}
(38) \quad D_1 ([\alpha^j] \kappa_{[r]}(\mathbf{u})) + D_2 ([\alpha^{j-1}] \kappa_{[r]}(\mathbf{u})) \\
= \left( (N-1)|\lambda^{[r]}| - b((\lambda^{[r]})^t) \right) [\alpha^j] \kappa_{[r]}(\mathbf{u}) \\
+ [\alpha^{j-1}] (A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)).
\end{aligned}$$

We recall that our goal is to prove that

$$(39) \quad [\alpha^j] \kappa_{[r]}(\mathbf{u}) = 0$$

for any  $0 \leq j \leq r - 2$ . We proceed by induction on  $j$ .

Consider the case  $j = 0$ . Since  $\kappa_{[r]}(\mathbf{u})$ ,  $A_1$  and  $A_2$  are polynomials in  $\alpha$ , Eq. (38) simplifies in this case to

$$D_1 f = \left( (N - 1) |\lambda^{[r]}| - b \left( \left( \lambda^{[r]} \right)^t \right) \right) f,$$

where  $f = [\alpha^0] \kappa_{[r]}(\mathbf{u})$ . Thanks to Proposition 4.1 we know that  $f$  satisfies the assumptions of Corollary 2.3 and hence it is equal to zero.

Now, we fix  $j \leq r - 2$ , and we assume that  $[\alpha^i] \kappa_{[r]}(\mathbf{u}) = 0$  holds true for all  $0 \leq i < j$ . We are going to show that it holds true for  $i = j$  as well.

Since  $[\alpha^{j-1}] \kappa_{[r]}(\mathbf{u}) = 0$  by the inductive hypothesis, Eq. (38) reads

$$\begin{aligned} D_1 ([\alpha^j] \kappa_{[r]}(\mathbf{u})) &= \left( (N - 1) |\lambda^{[r]}| - b \left( \left( \lambda^{[r]} \right)^t \right) \right) [\alpha^j] \kappa_{[r]}(\mathbf{u}) \\ &\quad + [\alpha^{j-1}] (A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)). \end{aligned}$$

First, we claim that  $[\alpha^{j-1}] A_1(\lambda^1, \dots, \lambda^r) = 0$ . Indeed, from the induction hypothesis, for each subset  $I$  with  $\emptyset \subsetneq I \subsetneq [r]$ , one has  $\kappa_I(\mathbf{u}) = O(\alpha^{|I|-1})$  and  $\kappa_{I^c}(\mathbf{u}) = O(\alpha^{|I^c|-1}) = O(\alpha^{r-|I|-1})$ . We then use Lemma 4.3 and write:

$$\begin{aligned} [\alpha^{j-1}] A_1(\lambda^1, \dots, \lambda^r) &= b \left( \lambda^{[r]} \right) [\alpha^{j-1}] \kappa_{[r]}(\mathbf{u}) \\ &\quad + \frac{1}{2} \sum_{\emptyset \subsetneq I \subsetneq [r]} [\alpha^{j-1}] \text{InEx}(\lambda^I, \lambda^{I^c}) \kappa_I(\mathbf{u}) \kappa_{I^c}(\mathbf{u}) = 0, \end{aligned}$$

since  $j - 1 < r - 2$ .

Similarly, one can prove that  $[\alpha^{j-1}] A_2(\lambda^1, \dots, \lambda^r) = 0$ . Indeed, using a similar argument as before, we have

$$\frac{1}{2} \sum_{1 \leq m \leq N} \sum_{\emptyset \subsetneq I \subsetneq [r]} \left( x_m \frac{\partial}{\partial x_m} \kappa_I(\mathbf{u}) \right) \left( x_m \frac{\partial}{\partial x_m} \kappa_{I^c}(\mathbf{u}) \right) = O(\alpha^{r-2}).$$

But, from Lemma 4.4, the left-hand side is  $A_2(\lambda^1, \dots, \lambda^r)$ . Since  $j - 1 < r - 2$ , we know that  $[\alpha^{j-1}] A_2(\lambda^1, \dots, \lambda^r) = 0$ , as wanted.

Above computations show that Eq. (38) simplifies to

$$D_1 f = \left( (N - 1) |\lambda^{[r]}| - b \left( \left( \lambda^{[r]} \right)^t \right) \right) f,$$

where  $f = [\alpha^j] \kappa_{[r]}(\mathbf{u})$ . Again, thanks to Proposition 4.1 we know that  $f$  satisfies assumptions from Corollary 2.3 and thus it is equal to zero, which finishes the proof.  $\square$

## 5. POLYNOMIALITY IN $b$ -CONJECTURE

**5.1. Cumulants and Young diagrams.** Consider a function  $F$  on Young diagrams and some diagrams  $\lambda^1, \dots, \lambda^r$ . Then we consider the family defined by



(recall that we use  $\oplus$  for entry-wise sum of partitions):

$$(40) \quad u_I = F \left( \bigoplus_{i \in I} \lambda^i \right).$$

**Definition 5.1.** We say that a function  $F$  on Young diagrams has the small cumulant property if, for any  $r \geq 1$  and for any partitions  $\lambda^1, \dots, \lambda^r$ , the above-defined family has the small cumulant property (in the sense of Proposition 3.3).

With this notation, the results of the previous sections can be restated as follows:

**Theorem 1.4:** For a fixed alphabet  $\mathbf{x}$ , the function  $\lambda \mapsto J_\lambda^\alpha(\mathbf{x})$  has the small cumulant property.

**Proposition 3.6:** The function  $\text{hook}_\alpha$  has the small cumulant property.

**Proposition 3.7:** The function  $\text{hook}_\alpha''$  has the small cumulant property.

**Corollary 3.4:** If  $F_1$  and  $F_2$  have the small cumulant property and take non-zero values, then so have  $F_1 \cdot F_2$  and  $F_1/F_2$ .

As a consequence, the function

$$\lambda \mapsto \frac{1}{\text{hook}_\alpha(\lambda) \text{hook}_\alpha''(\alpha)} J_\lambda^\alpha(\mathbf{x}) J_\lambda^\alpha(\mathbf{y}) J_\lambda^\alpha(\mathbf{z})$$

has the small cumulant property. We will use that later in this section.

*Remark.* Another consequence is that the function  $\lambda \mapsto \frac{J_\lambda^\alpha(\mathbf{x})}{\text{hook}_\alpha(\lambda)}$  also has the small cumulant property. We will not use this result here, but since this function is the standard  $P$ -normalization of Jack polynomials, we decided to mention it here.

**5.2. Cumulants and logarithm.** Let  $\mathbf{t} = (t_1, t_2, \dots)$  be an infinite alphabet of formal variables. We use the notation  $\mathbf{t}^\lambda = t_{\lambda_1} \cdots t_{\lambda_r}$ .

**Lemma 5.2.** Let  $F$  be a function on Young diagrams. Denote  $\kappa^F(\lambda^1, \dots, \lambda^r)$  the cumulant  $\kappa_{[r]}(\mathbf{u})$ , where  $\mathbf{u}$  is defined by Eq. (40). Then we have the following equality of formal power series in  $\mathbf{t}$ :

$$\log \sum_{\lambda} \frac{F(\lambda)}{\alpha^{\lambda_1} \prod_i m_i(\lambda^t)!} \mathbf{t}^{\lambda^t} = \sum_{r \geq 1} \frac{1}{r! \alpha^r} \sum_{(j_1, \dots, j_r)} \kappa^F(1^{j_1}, \dots, 1^{j_r}) t_{j_1} \cdots t_{j_r}.$$

*Proof.* Both sides expand as linear combinations of products

$$F_{\lambda^1, \dots, \lambda^s} := F(\lambda^1) \cdots F(\lambda^s) \mathbf{t}^{(\lambda^1)^t} \cdots \mathbf{t}^{(\lambda^s)^t},$$

where  $\lambda^1, \dots, \lambda^s$  are partitions. Fix some partitions  $\lambda^1, \dots, \lambda^s$ . The coefficient of  $F_{\lambda^1, \dots, \lambda^s}$  on the left-hand side is given by

$$(41) \quad \frac{(-1)^{s-1}}{s} \frac{s!}{|\text{Aut}(\lambda^1, \dots, \lambda^s)|} \prod_{h=1}^s \frac{1}{\alpha^{\lambda_h^1} \prod_i m_i((\lambda^h)^t)!}.$$

Here,  $|\text{Aut}(\lambda^1, \dots, \lambda^s)|$  denotes the number of permutations  $\sigma$  of size  $s$  such that  $\lambda^j = \lambda^{\sigma(j)}$  for all  $j \leq s$ .

The situation on the right-hand side is more intricate. First, rewrite it as

$$(42) \quad \sum_{r \geq 1} \frac{1}{r! \alpha^r} \sum_{(j_1, \dots, j_r)} \kappa^F(1^{j_1}, \dots, 1^{j_r}) t_{j_1} \cdots t_{j_r} \\ = \sum_{r \geq 1} \sum_{(j_1, \dots, j_r)} \sum_{\pi \in \mathcal{P}([r])} \frac{\mu(\pi, \{[r]\})}{r! \alpha^r} \prod_{B \in \pi} F \left( \bigoplus_{h \in B} 1^{j_h} \right) t_{j_1} \cdots t_{j_r}.$$

We are interested in which summation indices contribute to the coefficient of  $F_{\lambda^1, \dots, \lambda^s}$ , that is indices such that one has the following equality of the multisets

$$\left\{ \bigoplus_{h \in B} 1^{j_h}, B \in \pi \right\} = \{\lambda^h, 1 \leq h \leq s\}.$$

First,  $(j_1, \dots, j_r)$  should be a reordering of list of column lengths in  $\lambda^1, \dots, \lambda^s$ . If  $m'_i$  denotes the number of  $i$  in this list of column lengths, there are  $r! / (\prod_i m'_i!)$  such reordering and each gives the same contribution to the coefficient of  $F_{\lambda^1, \dots, \lambda^s}$ . We now suppose that we have fixed such a reordering  $(j_1, \dots, j_r)$ .

By definition, the number of columns of length  $i$  in  $\lambda^j$  is  $m_i((\lambda^j)^t)$ . Then the number of *ordered* set partitions  $(B_1, \dots, B_s)$  of  $[r]$  such that

$$\bigoplus_{b \in B_h} 1^{j_b} = \lambda^h \text{ for } 1 \leq h \leq s$$

is  $(\prod_i m_i!) / (\prod_{i,j} m_i((\lambda^j)^t)!)^s$ . Indeed, for each value  $i$ , one has to choose  $m_i((\lambda^1)^t)$  entries equal to  $i$  in the list  $(j_1, \dots, j_r)$  that go in  $B_1$ ,  $m_i((\lambda^2)^t)$  entries equal to  $i$  that go in  $B_2$ , and so on. This gives for each  $i$  a multinomial  $m_i! / (\prod_j m_i((\lambda^j)^t))$ , as claimed. But we want to count (unordered) set partitions and not ordered set partitions as above, so that we should divide by  $|\text{Aut}(\lambda^1, \dots, \lambda^s)|$ .

All these set partitions have  $s$  blocks so that the corresponding value of the Möbius function is  $\mu(\pi, \{[r]\}) = (-1)^{s-1} (s-1)!$ .

Finally, the coefficient of  $F_{\lambda^1, \dots, \lambda^s}$  in Eq. (42) is

$$(43) \quad \frac{r!}{\prod_i m_i!} \frac{\prod_i m_i!}{\prod_{i,j} m_i((\lambda^j)^t)} \frac{1}{|\text{Aut}(\lambda^1, \dots, \lambda^s)|} \frac{(-1)^{s-1} (s-1)!}{r! \alpha^r},$$

where  $r$  is the total number of columns in the  $\lambda^1, \dots, \lambda^s$ , that is  $r = \sum_h \lambda_1^h$ .

Comparing Eq. (41) and Eq. (43), we get our result.  $\square$

*Remark.* The statement and proof of this lemma are similar to the fact that cumulants can be alternatively defined as a sum over set partitions or as coefficients in the generating series of the logarithm of the moment generating series; see, e.g. [LS59, Eqs (3) and (II.c)].

**5.3. Conclusion.** We have now all the tools needed to prove the polynomiality in  $b$ -conjecture.

*Proof of Theorem 1.2.* Thanks to Corollary 2.5, it is enough to prove that  $h_{\mu,\nu}^\tau(\beta)$  has no pole in  $\alpha = 0$ , i.e. that  $h_{\mu,\nu}^\tau(\beta) = O(1)$ . From Eq. (1), this amounts to establish that

$$\log \left( \sum_{\tau \in \mathcal{P}} \frac{J_\tau^{(\alpha)}(\mathbf{x}) J_\tau^{(\alpha)}(\mathbf{y}) J_\tau^{(\alpha)}(\mathbf{z}) t^{|\tau|}}{\text{hook}_\alpha(\lambda) \text{hook}'_\alpha(\lambda)} \right) = O(\alpha^{-1}).$$

But, using Eq. (29), we see that this quantity is the left-hand side of Lemma 5.2 for

$$F(\lambda) = \frac{1}{\text{hook}_\alpha(\lambda) \text{hook}''_\alpha(\alpha)} J_\lambda^\alpha(\mathbf{x}) J_\lambda^\alpha(\mathbf{y}) J_\lambda^\alpha(\mathbf{z})$$

and  $t_1 = t_2 = \dots = t$ . It was observed at the end of Section 5.1 that this function  $F$  has the small cumulant property. Therefore, for any  $j_1, \dots, j_r$ , the cumulant  $\kappa^F(1^{j_1}, \dots, 1^{j_r})$  is  $O(\alpha^{r-1})$  and, thus, the right-hand side of Lemma 5.2 is  $O(\alpha^{-1})$ . This finishes the proof of the polynomiality.

The bound on the degree follows from the polynomiality and work of La Croix, see [La 09, Lemma 5.7 and Theorem 5.18].  $\square$

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